Chapter 5 Problems

1. a. Prove that

$$\sum_{n\leq x}\omega(n)=x\log\log x+b_1x+O\left(\frac{x}{\log x}\right).$$

2. i. Let p and q denote prime numbers. Explain why

$$\left(\sum_{p\leq\sqrt{x}}\frac{1}{p}\right)^2\leq\sum_{pq\leq x}\frac{1}{pq}\leq\left(\sum_{p\leq x}\frac{1}{p}\right)^2.$$

ii. Deduce that

$$\sum_{pq\leq x}\frac{1}{pq}=(\log\log x)^2+O(\log\log x).$$

iii. Deduce that

$$\sum_{n\leq x}\omega^2(n)=x\left(\log\log x\right)^2+O(x\log\log x),$$

the second result in Theorem 1 but now with equality, not an inequality.

3. We start examining the average of Ω with

$$\sum_{n\leq x}\Omega(n)=\sum_{n\leq x}\sum_{p^r|n}1=\sum_{p^r\leq x}\sum_{\substack{n\leq x\\ p^r|n}}1=\sum_{p^r\leq x}\left[\frac{x}{p^r}\right].$$

i. Show that

$$\sum_{n\leq x}\left(\Omega(n)-\omega(n)\right)=x\sum_{\substack{p\\ p^r\leq x}}\sum_{r\geq 2}\frac{1}{p^r}+O\left(\sum_{\substack{p\\ p^r\leq x}}\sum_{r\geq 2}1\right).$$

ii. Show that

$$\sum_{\substack{p\\ p^r\leq x}}\sum_{r\geq 2}1=\pi\left(\sqrt{x}\right)+O\left(x^{1/3}\right).$$

iii. Show that the tail end

$$\sum_{\substack{p\\ p^r>x}}\sum_{r\geq 2}\frac{1}{p^r}=O\left(\frac{1}{x^{1/4}}\right).$$

Hint. A bit of trick, but use the idea that for $p^r \geq x$ and $r \geq 2$ we have $x^{1/4} p^{3r/4} \leq p^r$.

iv. Combine all parts to deduce that

$$\sum_{n \leq x} (\Omega(n) - \omega(n)) = x \sum_p \frac{1}{p\,(p-1)} + O\big(x^{3/4}\big)\,.$$

v. Deduce that

$$\sum_{n \leq x} \Omega(n) = x \log\log x + b_2 x + O\left(\frac{x}{\log x}\right),$$

for some constant b_2.